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WAVE FORMATION IN A FILM FLOWING DOWN AN INCLINED PLANE IN THE PRESENCE OF PHASE CHANGE AND TANGENTIAL TENSION ON A FREE SURFACE

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It is known that the flow of thin layers of a viscous liquid is accompanied by wave phenomena. These phenomena in both linear and nonlinear formulations for a free falling film has been studied in many works [1–5]. Wave formation in joint flows of liquid and gas has been studied much less. For the case of a horizontal channel, the linear stability of film flow produced by gas flow is studied in [6–8], while for the case of a vertical plane, in [9, 10]. The presence of a transverse mass flux (for example, in condensation or evaporation) exerts a strong influence on wave formation; in the absence of tangential tensions on a free surface, this influence was studied in [11, 12].

The goal of this work is to study wave formation in film flow within the framework of an approach that takes into account several factors: the slope angle of the flow plane, the tangential tension on a free surface, and the phase change (condensation or evaporation). This will allow us to determine the role of various factors in wave formation and to gain further insight into the mechanism of wave instability.

1. Statement of the Problem. The flow pattern is given in Fig. 1. The fundamental state of the system is described by the velocity fields u_0 and v_0 in the liquid phase and U_0 and V_0 in the vapor phase and also by the function $h_0(x_*)$ (x_* is a large-scale variable along the flow plane). In what follows, we are interested in flow stability with respect to disturbances of the free surface $h = \widehat{h}\exp[i\alpha(x - ct)]$, where c is the unknown complex increment of buildup or damping; α is the real wave number ($\alpha = 2\pi/\lambda$, λ is the length of the disturbance wave).

Imposing the disturbance field upon the fundamental state and linearizing the original equations of motion, we obtain the Orr–Sommerfeld system of equations

$$\begin{gathered} i\alpha\mathrm{Re}[(u_0 - c)(f_{yy} - \alpha^2 f) - u_{0yy}f] = f_{yyyy} - 2\alpha^2 f_{yy} + \alpha^4 f, \\ u = -f_y\widehat{h}\exp[i\alpha(x - ct)], \quad v = i\alpha f\widehat{h}\exp[i\alpha(x - ct)]; \end{gathered} \tag{1.1}$$

$$\begin{gathered} i\alpha_g\mathrm{Re}_g[(U_0 - C)(F_{yy} - \alpha_g^2 F) - U_{0yy}F] = F^{\mathrm{IV}} - 2\alpha_g^2 F'' + \alpha_g^4 F, \\ U = -F_y\widehat{H}\exp[i\alpha_g(X - Ct)], \quad V = i\alpha_g F\widehat{H}\exp[i\alpha_g(X - Ct)] \end{gathered} \tag{1.2}$$

with the boundary conditions

$$f\Big|_{y=0} = \frac{df}{dy}\Big|_{y=0} = 0, \qquad F\Big|_{Y=\infty} = \frac{dF}{dY}\Big|_{Y=\infty} = 0, \qquad \sigma_{xy}^{(g)}\Big|_{y=h_0+h} = \sigma_{xy}^{(l)}\Big|_{y=h_0+h},$$

$$(\sigma_{yy}^{(g)} - \sigma_{yy}^{(l)})\Big|_{y=h_0+h} = -\frac{\sigma}{h_0}\frac{\partial^2 h}{\partial x^2}, \qquad \frac{\langle u_0\rangle}{U_\infty}u\Big|_{y=h_0+h} = U\Big|_{y=h_0+h},$$

$$\frac{\langle u_0\rangle}{U_\infty}v\Big|_{y=h_0+h} = V\Big|_{y=h_0+h}, \qquad v\Big|_{y=h_0+h} = \frac{d}{dt}(h_0 + h) + \frac{J}{\rho\langle u_0\rangle}.$$

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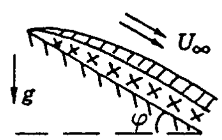

Fig. 1

Here and below the values denoted by capital letters correspond to the liquid phase, while those denoted by small letters, to the gas phase. The velocities in (1.1) and (1.2) are nondimensionlized by $\langle u_0\rangle$ and U_∞ ($\langle u_0\rangle$ is the velocity averaged over the liquid thickness, U_∞ is the velocity in the vapor core of the flow). The scale of thickness and length in (1.1) is the film thickness h_0, and in (1.2), the boundary-layer thickness δ_0; $\mathrm{Re} = u_0 h_0/\nu$ and $\mathrm{Re}_g = U_\infty \delta_0/\nu_g$ are Reynolds numbers (ν and ν_g are the kinematic viscosities of the liquid and the vapor); σ is the coefficient of surface tension; $\sigma_{xy}^{(g)}$, $\sigma_{yy}^{(g)}$, $\sigma_{xy}^{(l)}$, and $\sigma_{yy}^{(l)}$ are the components of the tension tensor in the vapor and the liquid [$\sigma_{xy} = \mu(\partial u/\partial y + \partial v/\partial x)$, $\sigma_{yy} = -p + 2\mu(\partial v/\partial y)$, μ is the dynamic viscosity in the liquid and the vapor, p is the pressure]; ρ is the liquid density; $J = -(\lambda/r)\partial T/\partial y\Big|_{y=h_0+h}$ is the transverse mass flux; λ is the thermal diffusivity; r is the heat of phase change.

To Eqs. (1.1) and (1.2) the equation for disturbance of the temperature field should be added. Here we restrict ourselves to the case of long-wave disturbances and use an approximation $\alpha/2\pi = h_0/\lambda \ll \min\{1, 1/(\mathrm{Re}\,\mathrm{Pr})\}$, (Pr is the Prandtl number for the liquid), which allows one to consider the temperature field "quasistationary" [12], i.e., $T = T_w + (T_s - T_w)y/(h_0 + h(x,t))$, where T_s is the temperature of the vapor phase and T_w is the wall temperature, $\Delta T = T_s - T_w$.

Performing expansion in the neighborhood of an undisturbed surface and using the dimensionless form, we bring the boundary conditions to the form

$$\begin{gathered}
f\Big|_{y=0} = \frac{df}{dy}\Big|_{y=0} = F\Big|_{y=\infty} = \frac{dF}{dY}\Big|_{Y=\infty} = 0, \quad f''\Big|_{y=1} + \alpha^2 f\Big|_{y=1} + \mathrm{Re}\Sigma^*\Big|_{Y=0} = 0,\\
-\widehat{p}\Big|_{y=1} + \frac{2i\alpha}{\mathrm{Re}}\frac{df}{dy}\Big|_{y=1} = -\alpha^2 T + (\Pi|_{Y=0} - G^*),\\
\frac{\langle u_0\rangle h_0}{U_\infty \delta}\Big(-\frac{df}{dy}\Big|_{y=1} + \frac{\mu_g}{\mu}\frac{dU_0}{dY}\Big|_{Y=0}\Big) = \Big(-\frac{dF}{dY}\Big|_{Y=0} + \frac{dU_0}{dY}\Big|_{Y=0}\Big),\\
\frac{\langle u_0\rangle h_0}{U_\infty \delta}\Big(i\alpha f\Big|_{y=1} + \frac{dv_0}{dy}\Big|_{y=1}\Big) = \Big(i\alpha_g F\Big|_{Y=0} + \frac{dV_0}{dY}\Big|_{Y=0}\Big),\\
i\alpha f\Big|_{y=1} + \frac{dh_0}{dx}\frac{df}{dy}\Big|_{y=1} = i\alpha(u_0\Big|_{y=1} - c) - \frac{dv_0}{dy}\Big|_{y=1} + \frac{1}{\mathrm{Re}\,\mathrm{Pr}\,\mathrm{Ku}},
\end{gathered} \tag{1.3}$$

where

$$\Sigma^*\Big|_{Y=0} = -K\Big(\frac{d^2F}{dY^2} + \alpha_g^2 F\Big)\Big|_{Y=0} + K\Big(\frac{\partial^2 U_0}{\partial Y^2} + \frac{\partial^2 V_0}{\partial X\,\partial Y}\Big)\Big|_{Y=0} - \frac{1}{\mathrm{Re}}\Big(\frac{\partial^2 u_0}{\partial y^2} + \frac{\partial^2 v_0}{\partial x\,\partial y}\Big)\Big|_{y=1};$$

$$K = \frac{\rho_g}{\rho}\Big(\frac{U_\infty}{\langle u_0\rangle}\Big)^2 \frac{h_0}{\delta}\frac{1}{\mathrm{Re}_g}; \qquad \widehat{p}\Big|_{y=1} = (u_0\Big|_{y=1} - c)f_y\Big|_{y=1} - u_{0y}f\Big|_{y=1} + \frac{i}{\alpha\mathrm{Re}}(f_{yyy} - \alpha^2 f_y)\Big|_{y=1}$$

(the expression for $\widehat{p}$ follows from the original linearized equations);

$$\Pi\Big|_{Y=0} = K\mathrm{Re}_g\Big[-\widehat{P}_g + \frac{2i\alpha_g}{\mathrm{Re}_g}\frac{dF}{dY}\Big]\Big|_{Y=0}; \quad \widehat{P}_g = (U_0 - C)F_Y - U_{0Y}F + \frac{i}{\alpha_g \mathrm{Re}_g}(F_{YYY} - \alpha_g^2 F_Y);$$

$$T = \sigma/(\rho\langle u_0\rangle^2 h_0); \qquad g^* = g(1 - \rho_g/\rho); \qquad G = \frac{g^*\cos\varphi\cdot h_0}{\langle u_0\rangle^2};$$

$$G^* = G - 2K \left.\frac{d^2V_0}{dY^2}\right|_{Y=0} + \frac{2}{\mathrm{Re}} \left.\frac{\partial^2 v_0}{\partial y^2}\right|_{y=1}; \qquad \mathrm{Pr} = \nu/\alpha; \qquad \mathrm{Ku} = r/C_p\Delta T.$$

For most cases that are interesting from a practical point of view, $\mu_g/\mu \ll 1$, $\langle u_0\rangle/U_\infty \ll 1$, and $h_0/\delta \ll 1$. The Orr–Sommerfeld problem for a vapor phase in this case is separated from the problem for liquid [6, 7] and is solved under the boundary conditions

$$F\Big|_{y=\infty} = \left.\frac{dF}{dY}\right|_{Y=\infty} = 0, \quad \left.\frac{dF}{dY}\right|_{Y=0} = \left.\frac{dU_0}{dY}\right|_{Y=0}, \quad F\Big|_{Y=0} = \frac{i}{\alpha_g}\left.\frac{dV_0}{dY}\right|_{Y=0}.$$

2. Solution of the Orr–Sommerfeld Equation in a Liquid. To solve the Orr–Sommerfeld equation in a liquid subject to the boundary conditions, we use the method of [1] and represent the solution as

$$f = \sum_{n=2}^{\infty} A_n y^n.$$

Substituting the general form of the stationary solution for problems of film flow $u_0(y) = 2y + \bar{u}(2y/3 - y^2)$ into (1.1), we obtain the recursive relation

$$\begin{gathered} n(n-1)(n-2)(n-3)A_n = (n-2)(n-3)\hat{p}A_{n-2} + (n-3)(n-4)\hat{q}A_{n-3} \\ + (n-4)(n-5)\hat{w}A_{n-4} - \alpha^2[\hat{r}A_{n-4} + \hat{q}A_{n-5} + \hat{w}A_{n-6}], \\ \hat{p} = 2\alpha^2 - i\alpha\,\mathrm{Re}\,c, \quad \hat{q} = 2i\alpha\mathrm{Re}(1+\bar{u}/3), \hat{w} = -i\alpha\mathrm{Re}\,\bar{u}, \quad \hat{r} = \alpha^2 - i\alpha\mathrm{Re}\,c - 2i\bar{u}\mathrm{Re}/\alpha. \end{gathered}$$

In what follows, we restrict ourselves to the case of $\alpha^2 \ll 1$ and $\alpha\mathrm{Re} < 1$, and neglecting the terms $O(\alpha\mathrm{Re})$ and $O(\alpha^2)$, we find

$$f = A_2(y^2 + \hat{p}y^4/12 + \hat{q}y^5/60) + A_3(y^3 + \hat{p}y^5/20 + \hat{q}y^6/60 + \hat{w}y^7/210). \tag{2.1}$$

After substitution of (2.1) into the boundary conditions (1.3), we have a system of three equations for A_2, A_3, and c:

$$\begin{gathered} 2i\alpha^2 A_2 - 6iA_3 = -\alpha^3\mathrm{Re}\,T + \alpha\mathrm{Re}(\Pi\big|_{Y=0} - G^*), \\ A_2[2 + i\alpha\mathrm{Re}(2/3 + 2\bar{u}/9 - c)] + A_3[6 + i\alpha\mathrm{Re}(1 + 2\bar{u}/15 - c)] = -\mathrm{Re}\Sigma^*, \\ A_2[1 + i\alpha\mathrm{Re}(1/30 + \bar{u}/90 - c/12) - (i/\alpha)dh_0/dx \cdot (2 + i\alpha\mathrm{Re}(1/6 + \bar{u}/18 - c/3))] \\ + A_3[1 + i\alpha\mathrm{Re}(1/30 + 2\bar{u}/315 - c/20) - (i/\alpha)dh_0/dx \cdot (3 + i\alpha\mathrm{Re}(1/5 + \bar{u}/30 - c/4))] \\ = 2 - \bar{u}/3 - c + (i/\alpha)[\partial v_0/\partial y\big|_{y=1} - 1/(\mathrm{Re}\,\mathrm{Pr}\,\mathrm{Ku})]. \end{gathered} \tag{2.2}$$

Solving the system of equations (2.2), with the accuracy specified above, we obtain a quadratic equation for the complex increment of buildup or damping:

$$\begin{gathered} \mathrm{Re}\Sigma^*[1 + i\alpha\mathrm{Re}(1/30 + \bar{u}/90 - c/12) - 2(i/\alpha)dh_0/dx] \\ + [2 - \bar{u}/3 - c + (i/\alpha)(\partial v_0/\partial y\big|_{y=1} - 1/(\mathrm{Re}\,\mathrm{Pr}\,\mathrm{Ku}))][2 + i\alpha\mathrm{Re}(2/3 + 2\bar{u}/9 - c)] \\ = i[-\alpha^3\mathrm{Re}T + \alpha\mathrm{Re}(\Pi\big|_{Y=0} - G^*)][-2/3 + 2(i/\alpha)dh_0/dx]. \end{gathered} \tag{2.3}$$

For further simplification and analysis of Eq. (2.3), it is necessary to have expressions for the components of the normal $\Pi\big|_{Y=0}$ and tangential $\sum^*$ tensions on the side of the vapor phase.

3. Method of Solving the Orr–Sommerfeld Equation in the Vapor Phase. The disturbance field in the vapor phase is divided into three regions in the same way as in the problem [6] of gas flow along a wavy wall: a) a "viscous sublayer" region with thickness δ_f, where the viscosity forces predominate; b) a boundary layer region with thickness δ_0, where the velocity profile pattern $U_0(y)$ is significant; and c) the flow core, where $U_0(y) = \mathrm{const}$.

The general solution to Eq. (1.2) is represented as a superposition of a "viscous" and "nonviscous" solutions subject to the boundary conditions.

We assume that $\delta_f \ll \delta_0$ and in the viscous sublayer the velocity profile is linear. The critical layer $[U_0(y_*) = C]$ is assumed to lie within the viscous sublayer and this is true if $U_0'(0)\delta_f \gg C$. The value of C for the problem in the vapor phase is assumed to be zero. In this case, the viscous solution is written in the form [13]

$$F_\nu = \int_\infty^\eta d\eta' \int_\infty^{\eta'} \eta''^{1/2} H_{1/3}^{(1)}[2(i\eta'')^{3/2}/3]\, d\eta'', \qquad \eta = (\alpha_g \mathrm{Re}_g U_0'(0))^{1/3} Y \tag{3.1}$$

($H_{1/3}^{(1)}$ is a Hankel function). Solution (3.1) satisfies the damping condition at large Y.

Outside of the viscous layer, taking into account that $\alpha_g \mathrm{Re}_g \gg 1$, we have $U_0(Y)(\Phi_{YY} - \alpha_g^2\Phi) - U_{0YY}\Phi = 0$ for determination of the "nonviscous" solution.

In the flow core, $\Phi = C\exp[-\alpha_g Y]$, and, to solve the problem in the boundary layer, we use the following approximation of the velocity profile [6]:

$$U_0(Y) = \begin{cases} \sin(U_0'(0)Y), & Y \leqslant \pi/(2U_0'(0)), \\ 1, & Y \geqslant \pi/(2U_0'(0)). \end{cases}$$

The velocity profile in the vapor phase can be more properly taken into account only in considering a particular problem and, as shown in [6], this leads to the emergence of small correction factors in the corresponding formulas for Σ^* and $\Pi\big|_{Y=0}$.

Sewing together the solutions in the boundary layer and in the flow core and making use of the boundary conditions at the interface, we obtain a system of equations for the quantities B and M:

$$\Phi = B[\cos(LY) + (A/B)\sin(LY)], \quad Y \leqslant \pi/(2U_0'(0)), \quad L^2 = (U_0'(0))^2 - \alpha_g^2,$$
$$\frac{A}{B} = \frac{1 - \dfrac{\alpha_g}{L}\cot\left(\dfrac{\pi}{2}\dfrac{L}{U_0'(0)}\right)}{\cot\left(\dfrac{\pi}{2}\dfrac{L}{U_0'(0)}\right) + \dfrac{\alpha_g}{L}}, \quad \Phi(0) + MF_\nu(0) = \frac{i}{\alpha_g}\frac{dV_0}{dY}\bigg|_{Y=0}, \quad \Phi'(0) + MF_\nu'(0) = \frac{dU_0}{dY}\bigg|_{Y=0}. \tag{3.2}$$

The given system differs from a similar system in [6], where the problem of gas flow along a weakly sinuous wall was considered, in the nonzero right side of the penultimate equation, which is due to the transverse mass flux.

After solving Eqs. (3.2), it is easy to find expressions for the tension components:

$$\Sigma^*\big|_{Y=0} = K\frac{F_\nu''}{F_\nu}\bigg|_{\eta=0}\left(B - \frac{i}{\alpha_g}\frac{dV_0}{dY}\bigg|_{Y=0}\right) + K\left(\frac{\partial^2 U_0}{\partial Y^2} + \frac{\partial^2 V_0}{\partial X \partial Y}\right)\bigg|_{Y=0} - \frac{1}{\mathrm{Re}}\left(\frac{\partial^2 u_0}{\partial y^2} + \frac{\partial^2 v_0}{\partial x\, \partial y}\right)\bigg|_{y=1},$$
$$\Pi\big|_{Y=0} = K\mathrm{Re}_g B\frac{dU_0}{dY}\bigg|_{Y=0}, \qquad B = \frac{U_0'(0) - \dfrac{i}{\alpha_g}\dfrac{F_\nu'}{F_\nu}\bigg|_{\eta=0}\dfrac{dV_0}{dY}\bigg|_{Y=0}}{A\dfrac{L}{B} - \dfrac{F_\nu'(0)}{F_\nu(0)}}. \tag{3.3}$$

The ranges of validity of all the above assumptions are analyzed in detail in [6] and remain true for our case. In the deduction of (3.3) and in what follows, we take into account that $F_\nu'(0)/F_\nu(0) = -1.288m\exp[\pi i/6]$, $m = (\alpha_g\mathrm{Re}_g U_0'(0))^{1/3}$, $F_\nu''(0)/F_\nu(0) = 1.372m^2\exp[\pi i/3]$, $F_\nu'''(0)/F_\nu(0) = -im^3$, as is easy to obtain from (3.1), using the integral representation of the Hankel function and changing the integration order.

Expressions (3.3) are still rather complex for further analysis. Further simplification is achieved with the supplementary (and true for most cases of practical interest) assumption [6] that $\alpha_g \ll U_0'(0)$, which is equivalent to the condition $\alpha_g/(C_f\mathrm{Re}) \ll 1$ (C_f is the friction coefficient for vapor).

In this case, the expression for B takes the form

$$B = \frac{\alpha_g}{U_0'(0)} \, \frac{1 + 1.288 \dfrac{im}{\alpha_g U_0'(0)} \dfrac{dV_0}{dY}\Big|_{Y=0} \exp\left[\dfrac{i\pi}{6}\right]}{1 + 1.288m \dfrac{\alpha_g}{(U_0'(0))^2} \exp\left[\dfrac{i\pi}{6}\right]}.$$

Let us analyze the order of quantities that enter in the denominator and numerator of this relation:

$$\frac{m\alpha_g}{(U_0'(0))^2} = \frac{2}{C_f} \frac{\alpha^{4/3}\beta^2}{\mathrm{Re}^{2/3}} \frac{1}{(2-4\bar{u}/3)^{2/3}} \ll 1, \qquad \beta = (\rho_g/\rho)^{1/3}(\nu_g/\nu)^{2/3} \ll 1, \quad \alpha \ll 1,$$

$$\frac{m}{\alpha_g U_0'(0)} \frac{dV_0}{dY}\Big|_{Y=0} = \frac{1}{3\alpha} \frac{\partial \bar{u}}{\partial x} (\alpha \mathrm{Re})^{1/3} \frac{1}{(2-4\bar{u}/3)^{2/3}} \left(\frac{\rho_g}{\rho}\right)^{2/3} \left(\frac{\nu_g}{\nu}\right)^{1/3} \ll 1, \quad \alpha\mathrm{Re} \ll 1, \qquad \frac{\partial \bar{u}}{\partial x} \approx \frac{1}{\mathrm{Pr\,Ku}} \ll 1.$$

Here α and Re correspond to the liquid film and use is made of $C_f = 2\nu_g U_{\mathrm{d}}'(0)/U_\infty^2$, $\mu_g U_{\mathrm{d}}'(0) = \mu du_{0\mathrm{d}}/dy\big|_{y=h0} = \mu\langle u_0\rangle(2-4\bar{u}/3)/h_0$, $\partial V_0/\partial Y = -\partial U_0/\partial X$, $U_0\big|_{Y=0} = \langle u_0\rangle u\big|_{y=1}/U_\infty$ (quantities with subscript d are dimensional). With allowance for these estimates, the formula for B has the form

$$B = \frac{\alpha_g}{U_0'(0)}.$$

Estimating similarly the order of the quantities contained in the expression for $\Sigma^*\big|_{Y=0}$, we obtain the final formulas for the tension components with the accuracy specified above:

$$\Pi\Big|_{Y=0} = \frac{2\alpha}{C_f\mathrm{Re}}(2 - \frac{4\bar{u}}{3}),$$
$$\Sigma^*\Big|_{Y=0} = \frac{2\beta}{C_f} \frac{\alpha^{5/3}}{\mathrm{Re}^{4/3}}\left(2-\frac{4\bar{u}}{3}\right)^{2/3} 1.372 \exp\left[\frac{\pi i}{3}\right] + K\left(\frac{\partial^2 U_0}{\partial Y^2} + \frac{\partial^2 V_0}{\partial X \partial Y}\right)\Big|_{Y=0} - \frac{1}{\mathrm{Re}}\left(\frac{\partial^2 u_0}{\partial y^2} + \frac{\partial^2 v_0}{\partial x \partial y}\right)\Big|_{y=1}. \tag{3.4}$$

4. Results. Using Eqs. (2.3) and (3.4) and representing $c = c_r + i\gamma$ we find expressions for the increment of buildup (damping) and for c_r with the specified accuracy:

$$\gamma = \frac{\alpha\mathrm{Re}}{2}\left[\frac{16}{15}\bar{u}\left(1+\frac{\bar{u}}{3}\right) + \frac{2}{3}(\Pi\big|_{Y=0} - G - \alpha^2 T) + \frac{\Sigma_i}{\alpha}\Big|_{Y=0} - \frac{4\bar{u}}{\alpha^2\mathrm{Re}}\frac{dh_0}{dx} + \frac{2}{\alpha^2\mathrm{Re}}\left(\frac{\partial v_0}{\partial y}\Big|_{y=1} - \frac{1}{\mathrm{RePrKu}}\right)\right],$$
$$c_r = 2 + \frac{2}{3}\bar{u}, \quad \Pi\Big|_{Y=0} = \frac{2\alpha}{C_f\mathrm{Re}}\left(2-\frac{4\bar{u}}{3}\right), \quad \Sigma_i\Big|_{Y=0} = \frac{\beta}{C_f}\frac{\alpha^{5/3}}{\mathrm{Re}^{4/3}}\left(2-\frac{4\bar{u}}{3}\right)^{2/3} 1.372\sqrt{3}, \quad \beta = \left(\frac{\rho_g}{\rho}\right)^{1/3}\left(\frac{\nu_g}{\nu}\right)^{2/3}. \tag{4.1}$$

The disturbances with $\gamma_i < 0$ are damping, while those with $\gamma_i > 0$ are growing. From (4.1), it follows that gravity, surface tension, and positive transverse mass flux (in the case of condensation) always have a stabilizing effect (negative contribution to the expression for γ_i). The vapor flow, on the one hand, decreases the average thickness of the film, leads to a more intense transverse mass flux, and, in the case of condensation, stabilizes the film. On the other hand, the value of Σ_i is always positive and has a destabilizing effect. Thus, we can speak about the overall effect of the vapor flow only after considering a particular problem.

Let us now analyze the various limiting cases.

(1) *Liquid Film Flowing Down Freely under Gravity.* In this case, $dh_0/dx = \Pi\big|_{Y=0} = \Sigma_i\big|_{Y=0} = 0$, and the stationary solution has the form

$$u_0(y) = \frac{g^*\sin(\varphi)}{\nu} h_0^2\left[\frac{y}{h_0} - \frac{y^2}{2h_0^2}\right], \qquad \mathrm{Re} = \frac{q_0}{\nu} = \frac{g^*\sin(\varphi)h_0^3}{3\nu^2},$$

whence, using the definitions of the dimensionless quantities, we obtain $\bar{u} = 3/2$, $G = 3\cot(\varphi)/\mathrm{Re}$, $T = (3\mathrm{Fi}/\sin(\varphi))^{1/3}/\mathrm{Re}^{5/3}$, $\mathrm{Fi} = (\sigma/\rho)^3/g^*\nu^4$.

From (4.1) follows the equation of a neutral curve ($\gamma = 0$) $G + \alpha^2 T = 18/5$, which does not have solutions under the condition $G > 18/5$. This leads to the well-known stability criterion for film flow on an inclined plane [1]: $\cot(\varphi) > 6\mathrm{Re}/5$.

(2) *Liquid Film Flowing Down Freely under Gravity in the Presence of a Transverse Mass Flux.* In this case $\Pi\Big|_{Y=0} = \Sigma_i\Big|_{Y=0} = 0$. The stationary solution is similar to that in item (1), and in addition,

$$\frac{dh_0}{dx} = \frac{1}{3\mathrm{Re\,Pr\,Ku}}, \qquad \frac{\partial v_0}{\partial y}\Big|_{y=1} = -\frac{1}{\mathrm{Re\,Pr\,Ku}}.$$

From (4.1), the following equation is obtained for the neutral curve:

$$\alpha^4 \mathrm{Re}T + \alpha^2\mathrm{Re}\Big(G - \frac{18}{5}\Big) + \frac{9}{\mathrm{Re\,Pr\,Ku}} = 0,$$

where $G = 3\cot(\varphi)/\mathrm{Re}$ and $T = (3\mathrm{Fi}/\sin(\varphi))^{1/3}/\mathrm{Re}^{5/3}$.

The critical Reynolds number at which wave formation starts is determined from the equation

$$\mathrm{Re}_c = \frac{5}{6}\cot(\varphi) + \frac{5}{3(\mathrm{Re}_c)^{5/6}}\Big[\Big(\frac{3\mathrm{Fi}}{\sin(\varphi)}\Big)^{1/3}\frac{1}{\mathrm{Pr\,Ku}}\Big]^{0.5} \tag{4.2}$$

and for $\varphi = \pi/2$ exactly corresponds to the one found in [11]. For the other values of the angle φ, the critical number of wave formation should be determined numerically from Eq. (4.2), which differs from the conclusions of [11], where a simplified expression $\mathrm{Re}_c = 5\cot(\varphi)/6$ for $\varphi \neq \pi/2$ is obtained.

(3) *Liquid Film Flowing Down under Gravity in the Presense of Tangential Tension on a Free Surface.* In this case, $dh_0/dx = \partial v_0/\partial y = 0$, and the stationary solution takes the form

$$u_0(y) = \frac{g^*\sin(\varphi)}{\nu}h_0^2\Big[\frac{y}{h_0} - \frac{y^2}{2h_0^2}\Big] + \frac{\tau_g y}{\mu}, \qquad \tau_g = \frac{1}{2}\rho_g U_\infty^2 C_f.$$

As a system of basic parameters, it is convenient to select the slope angle of the flow plane φ, $\mathrm{Re}^* = g^*h_0^3/(3\nu^2)$ and the velocity in the gas flow core U_∞. Then, the dimensionless criterion in (4.1) is expressed in the form

$$\mathrm{Re} = \mathrm{Re}^*\sin(\varphi) + \tau_g^*(\mathrm{Re}^*)^{2/3}, \quad \tau_g^* = 3^{2/3}\rho_g U_\infty^2 C_f/(4\rho(\nu g^*)^{2/3}), \quad \bar{u} = 1.5(1-\tau_*),$$

$$\tau_* = 1/(1+\sin(\varphi)\mathrm{Re}^{*1/3}/\tau_g^*), \quad G = 3\cos(\varphi)(\tau_*/\tau_g^*)^2/(\mathrm{Re}^*)^{1/3}, \quad T = (3\mathrm{Fi})^{1/3}(\tau_*/\tau_g^*)^2/\mathrm{Re}^*.$$

From Eq. (4.1) follows the system of equations for the critical number of wave formation Re_c^* ($\gamma = 0$, $\partial\gamma/\partial\alpha = 0$):

$$\frac{1}{2}\Pi_r\Big|_{Y=0} + G - 2\alpha^2T - \frac{8}{5}\bar{u}\Big(1+\frac{\bar{u}}{3}\Big) = 0, \quad \Pi_r\Big|_{Y=0} - 2\alpha^2T + \frac{\Sigma_i}{\alpha}\Big|_{Y=0} = 0.$$

Solving this system, after a number of conversions we obtain a transcendental equation for Re_c^*:

$$\Big(\frac{1}{\mathrm{Re}_c^*}\Big)^{1/3} = \frac{K}{\omega^{2/3}}(\omega+\Phi_1)^{1/3}(-3\omega+\Phi_1),$$

$$\Phi_1 = \Big[\omega^2 + \cos(\varphi) - \frac{2}{5}\sin(\varphi)(\mathrm{Re}_c^*)^{2/3}(K^3K_1\omega + 3\sin(\varphi)(\mathrm{Re}_c^*)^{1/3})\Big]^{1/2}, \tag{4.3}$$

$$K_1 = 64(2/3)^{0.5}\beta_1^3(3\mathrm{Fi})^{1/6}, \quad K = (6C_f)^{1/3}/(4\beta_1), \quad \omega = \frac{\rho_g}{8\rho}U_\infty^2\sqrt{\frac{2\rho}{\sigma g^*}}, \quad \beta_1 = 3^{0.5}1.372\beta/2.$$

At $\varphi = 0$, Eq. (4.3) goes over into the equation of work [7].

The results of numerical analysis of Eq. (4.3) for the air–water system are given in Figs. 2 and 3. Note that the friction coefficient C_f should be determined by solution of the stationary problem. Experiments [7], that studied the case of a horizontal channel ($\varphi = 0$) in an air–water system has shown that the value of C_f differs only slightly from this value for a dry channel and can be calculated from the corresponding formulas, depending on the geometric parameters of the channel. In Figs. 2 and 3, curves 1–4 correspond to $C_f = 3\cdot10^{-3}$ and curves 1′–4′, to $C_f = 6\cdot10^{-3}$ (the numerical values of C_f are taken from [7]).

From an additional analysis of Eq. (4.1), after reducing the system of dimensionless quantities to our case, it is easy to deduce that always $\partial\gamma/\partial\omega > 0$. Hence it follows that the results of calculation of Eq. (4.3)

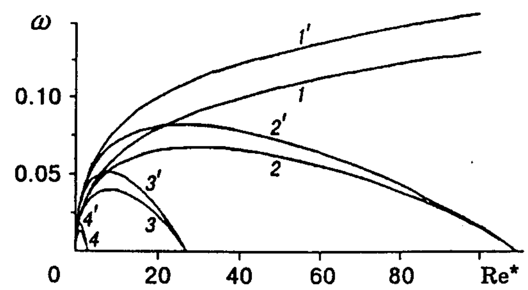

Fig. 2

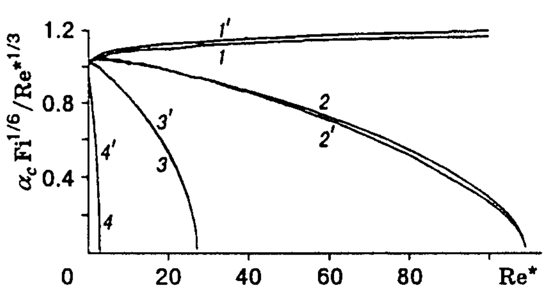

Fig. 3

are easy to represent as a set of the curves $\omega_c(\mathrm{Re}^*)$ for different φ, and in the range of $\omega > \omega_c$ we have rising disturbances, i.e., instability. In Figs. 2 and 3, curves 1, 1′–4, and 4′ are calculated at $\varphi = 0$, 5, 10, and 30°. For $\varphi \neq 0$, the curves in Fig. 2 bound the corresponding stability region in the plane (ω, Re^*), whose size decreases rapidly with increasing slope angle. The extreme right boundary point of the stability region corresponds to the critical number of wave formation in a film falling down freely $(5\cot(\varphi)/(6\sin(\varphi))$.

The results are conveniently illustrated by the numerical values of the velocity: $U_\infty = 1.71$, 2.42, 2.96, 3.42, 3.83, and 4.19 m/sec for $\omega = 0.025$, 0.05, 0.075, 0.1, 0.125, and 0.15.

Figure 3 shows wave numbers for disturbances at the boundary of the stability region. For an air–water system, the film number is rather large ($\mathrm{Fi} \sim 10^{11}$) and the main assumption of the present work (long waves) holds well, as follows from Fig. 3.

(4) *Wave Formation in Moving-Vapor Condensation along a Vertical Wall.* In this case, the stationary solution is determined by the system of equations [14]

$$u_0(y) = \frac{g^* h_0^2}{\nu}\left[\frac{y}{h_0} - \frac{y^2}{2h_0^2}\right] + \frac{\tau_g y}{\mu}, \quad \frac{1}{\nu}\left[\frac{h_0}{2}\frac{d}{dx}(u_s h_0)\right] + \frac{g^*}{\nu^2}\left[\frac{h_0^3}{4}\frac{dh_0}{dx}\right] = \frac{1}{\mathrm{Pr\,Ku}},$$

$$\frac{d}{dx}\left[\frac{h_0(U_\infty - u_s)^2(2U_\infty + 3u_s)}{u_s - g^* h_0^2/2\nu}\right] + \frac{15}{2}R^2\nu\left[\frac{(U_\infty - u_s)}{h_0}\frac{1}{\mathrm{Pr\,Ku}} - \frac{u_s - g^* h_0^2/2\nu}{h_0}\right] = 0, \tag{4.4}$$

$$\tau_g = \frac{\mu}{h_0}\left(u_s - \frac{g^* h_0^2}{2\nu}\right) = 0.5\rho_g U_\infty^2 C_f.$$

Here u_s is the velocity on the free film surface; $R^2 = \mu\rho/(\mu_g\rho_g)$.

Following [14] and sewing together the asymptotic solutions of Eqs. (4.4) at small and large x, we represent the solution as

$$\bar h_0^2 = \frac{\eta}{(1+\eta/4)^{1/2}}, \qquad \bar u_s = 4\left(1 + \frac{\eta}{16}\right)^{1/2}, \qquad \eta = (gx/U_\infty^2)\mathrm{Pr\,Ku}/\chi^4,$$
$$\bar h_0^2 = (gh_0^2/\nu U_\infty)\mathrm{Pr\,Ku}/\chi^2, \qquad \bar u_s = (u_s/U_\infty)\mathrm{Pr\,Ku}/\chi^2, \qquad \chi = 0.45(1.2 + \mathrm{PrKu}/R)^{1/3}. \tag{4.5}$$

The system of basic parameters and the expressions for Re, G, T, and $\bar u$ are similar to those in item (3) and after transformation the fundamental equation (4.1) for stability takes the form

$$\gamma = \frac{\alpha}{3\mathrm{Re}^{*1/3}}\frac{4\bar h_0^2}{\bar h_0^2 + 6\bar u_s}\left[4\alpha_1\omega - \frac{\alpha_1^2}{6\mathrm{Re}^{*1/3}} + \frac{18}{5}\mathrm{Re}^{*4/3}\frac{\bar u_s + \bar h_0^2/2}{\bar h_0^2}\right.$$
$$\left. + \frac{2^{1/3}3^{4/3}\alpha_1^{2/3}\omega^{2/3}(\mathrm{PrKu})^{2/3}}{\chi^{4/3}}\beta_1\left(\frac{\rho_g}{\rho}\right)^{1/3}\frac{\mathrm{Re}^{*1/9}}{(\bar h_0^2(\bar u_s - \bar h_0^2/2))^{1/3}} - \frac{Z}{\alpha_1^2}\right],$$
$$Z = \frac{54(3\mathrm{Fi})^{1/3}}{\mathrm{Re}^{*2/3}\mathrm{Pr\,Ku}}\left[1 + \frac{1}{6}\frac{d\bar h_0^2}{d\eta}(\bar h_0^2/2 - \bar u_s)\right],$$

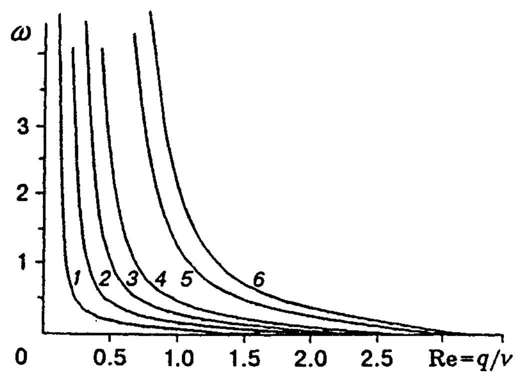

Fig. 4

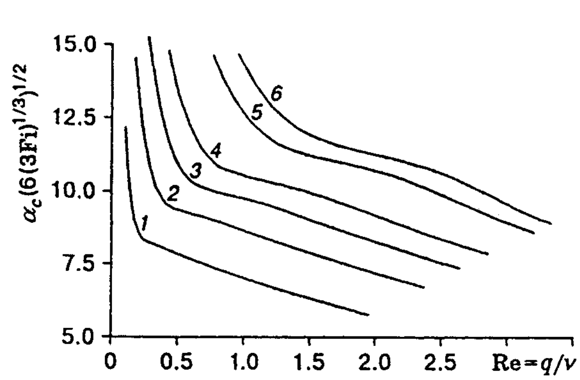

Fig. 5

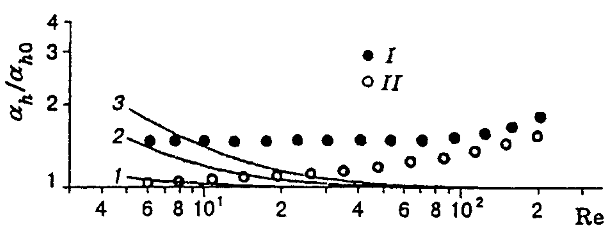

Fig. 6.

where $\bar{h}_0$, $\bar{u}_s$, and $d\bar{h}_0/d\eta$ can be implicitly expressed in terms of Re^*, ω, and PrKu by means of formulas (4.5); $\alpha_1 = \alpha(6(3\mathrm{Fi})^{1/3})^{1/2}$.

We can obtain a system of equations for the critical number of wave formation Re_c^* ($\gamma = 0$, $\partial\gamma/\partial\alpha = 0$) and solve it numerically. The results for the condensation of moving Freon-21 vapor (in the curve of saturation at $T_s = 60°$C) are given in Fig. 4 and 5. Curves 1–6 correspond to temperature differences of 5, 10, 15, 20, 30, and 35°C. The regions of instability lie to the right of the curves in Fig. 4, i.e., at such values of the parameters ω, Re, and ΔT, rising disturbances take place. Here we note that Re in Figs. 4 and 5 is constructed for the flow rate. The extreme right boundary point of the stability region corresponds to the critical number of the wave formation of a film flowing down freely in the presence of a phase change $[5[(3\mathrm{Fi})^{1/3}/\mathrm{PrKu}]^{0.5}/3]^{6/11}$ [see item (2)].

To illustrate the results, we give numerical velocity values: U_∞ = 0.52, 0.73, 0.9, 1.04, 1.47, and 1.80 m/sec for ω = 0.25, 0.5, 0.75, 1.0, 2.0, and 3.0.

In Fig. 5, the wave numbers for disturbances at the boundary of the stability region are presented. The notation of the curves is as in Fig. 4. For Freon-21, $(6(3\mathrm{Fi})^{1/3})^{1/2} \approx 57.8$, and the main assumption of the present work (long waves) holds well. As follows from the results, wave formation starts at small Reynolds numbers, despite the stabilizing effect of the transverse mass flux. In [15], experimental data were obtained on the integral heat-release coefficient α_h in the condensation of the moving vapor in a wide range of vapor temperatures, temperature differences, and flow rates in the condensate film [points I in Fig. 6 are taken from [15], and generalize the experimental data at $(\chi_{ex})^4\mathrm{Fr}/(\mathrm{PrKu}) = 0.1$ $(\pm 20\,\%)$, $\alpha_{h0} = 4\lambda(g/3\nu^2)^{1/3}/(3\mathrm{Re}^{1/3})$ is the integral heat-release coefficient during condensation of immovable vapor]. Points II in Fig. 6 are taken from [16], and generalize the experimental data on condensation of immovable vapor. The deviation of points II from the curve $\alpha_h/\alpha_{h0} = 1$ in the case of condensation of immovable vapor indicates the presence of waves, beginning from the smallest Reynolds numbers for the condensate film. For the moving vapor, we were unable to obtain an explicit theoretical relation between the integral heat-release coefficient ($\alpha_h = (1/L)\int_0^L q\,dx/\Delta T =$

$(\lambda/L)\int\limits_0^L dx/h_0)$ and the flow rate in the condensate film. Sewing together the asymptotic relations for this coefficient at small and large x, we obtain

$$\frac{\alpha_h}{\alpha_0}=\frac{3^{4/3}[1+(æ^{3/4}\mathrm{Re}/9)^{5/3}]^{2/5}}{(æ^{3/4}\mathrm{Re})^{2/3}},\qquad æ=\left(\frac{\nu g}{U_\infty^3}\right)^{2/3}\left(\frac{\mathrm{Pr\,Ku}}{\chi^2}\right)^2. \tag{4.6}$$

Formula (4.6) describes well the corresponding numerical calculation using Eq. (4.5) (accuracy of $\leqslant 0.5\%$ in a wide range of parameters). Note that data on heat release should be well generalized in the variables (α_h/α_0, $æ^{3/4}\mathrm{Re}$), which can be useful for data processing. Curves 1–3 in Fig. 6 correspond to calculations using Eqs. (4.6) for Freon-21 at $\Delta T = 1$, 5, and 10°C and the vapor velocity U_∞ was calculated from the experimental conditions (points I) $(\chi_{ex})^4\mathrm{Fr}/(\mathrm{PrKu}) = 0.1$. As follows from Fig. 6, the experimental curve begins to deviate from the theoretical curve for the smallest Reynolds numbers, which is obviously indicative of the presence of waves and is in agreement with the results on stability obtained in this work.

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